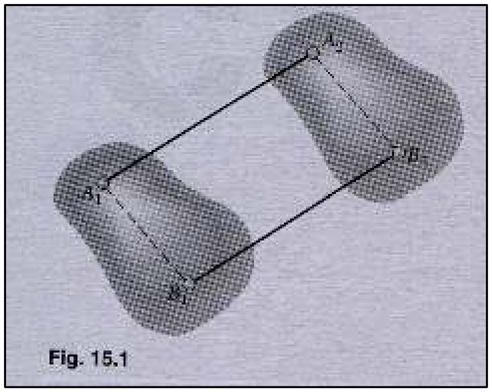
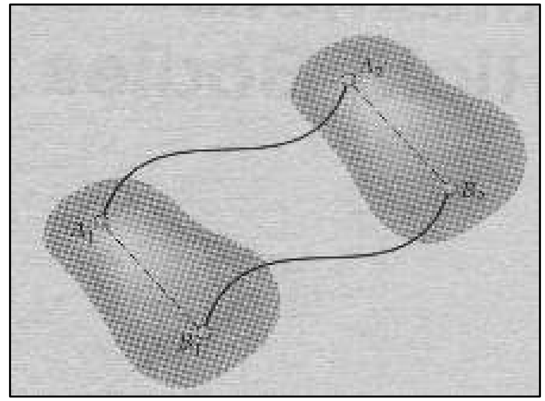
# Kinematics of rigid bodies

relations between time and the positions, velocities, and accelerations of the particles forming a rigid body.



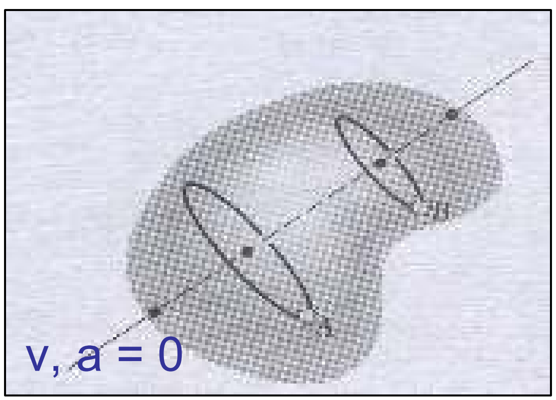

Rectilinear translation – parallel straight paths

**(1)**



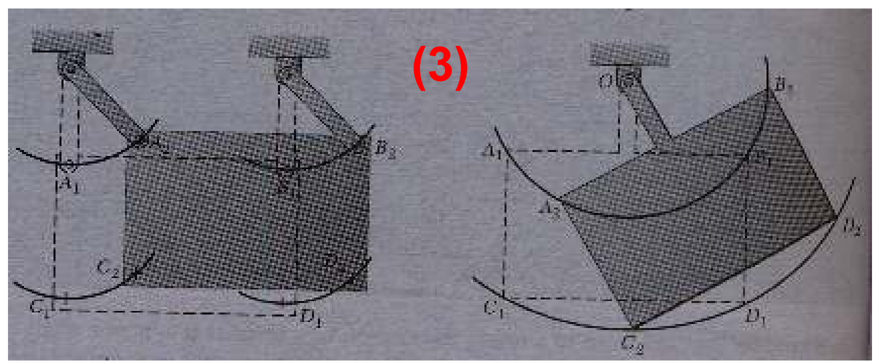
Curvilinear translation

**(2)**




Rotation about a fixed axis

**(3)**

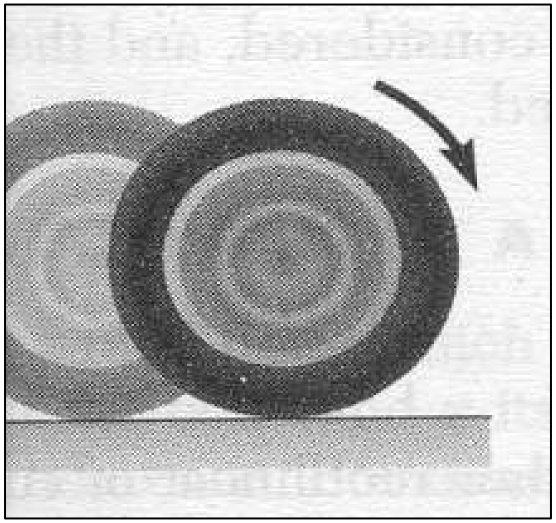


Parallel circles     concentric circles

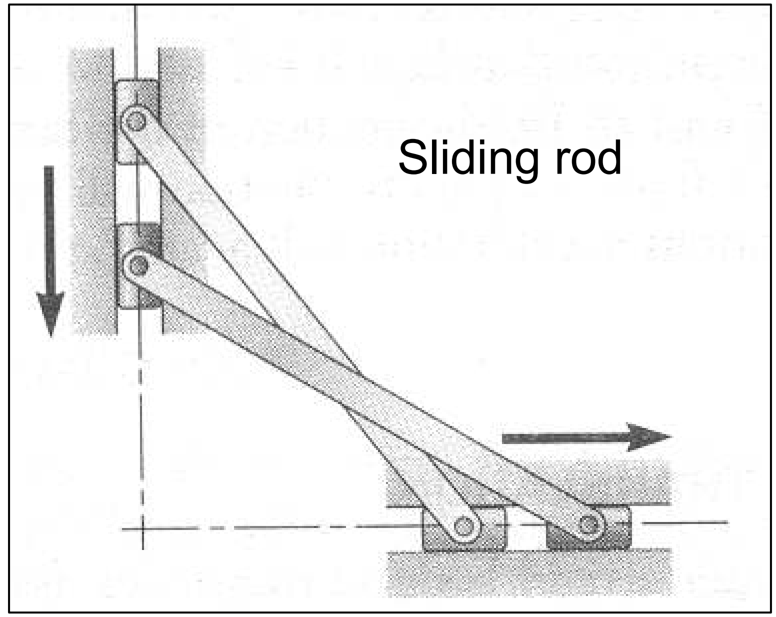
Curvilinear translation     rotation

Plane motion

**(4)**

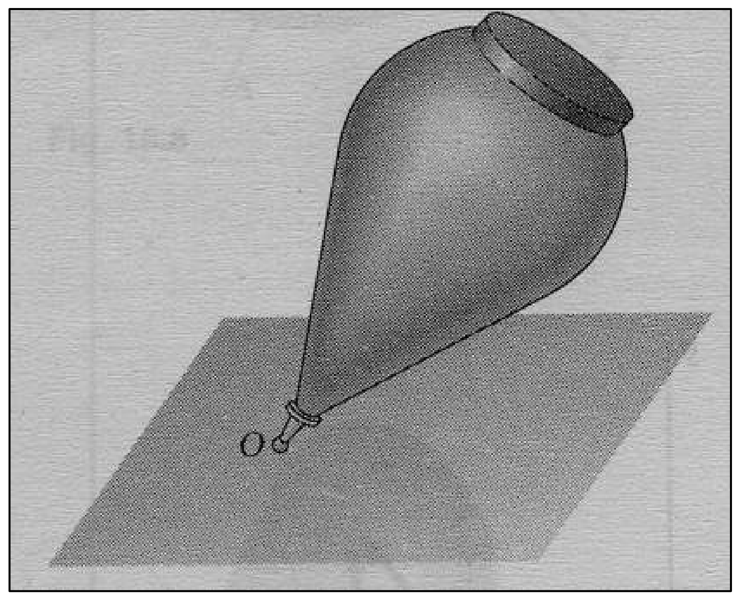


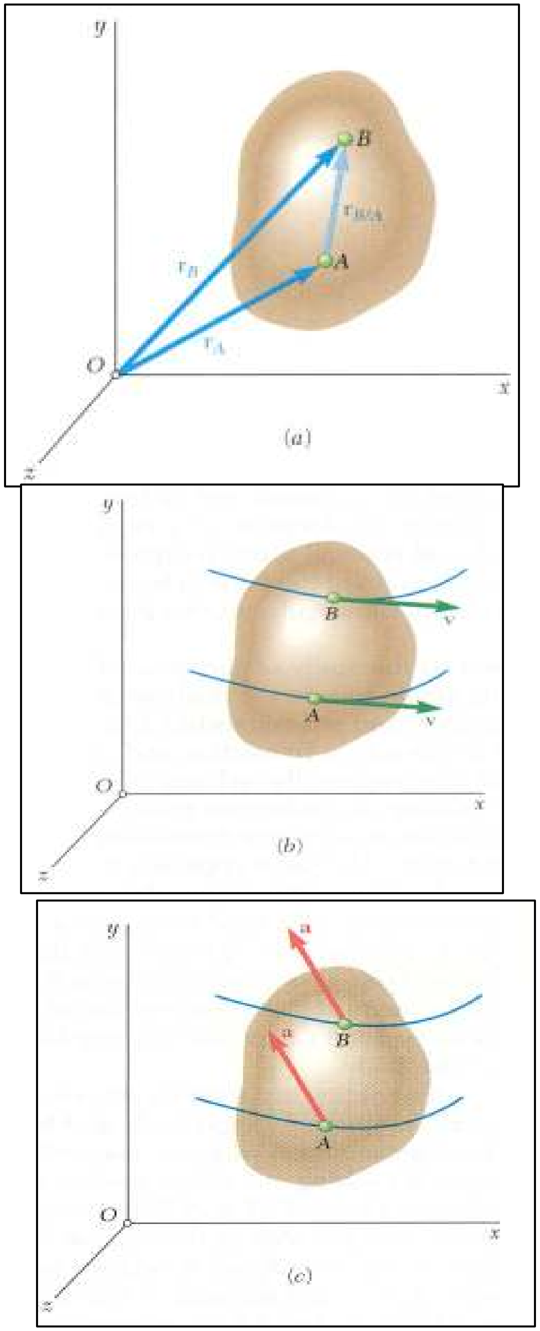
Rolling wheel




General Plane motion – neither rotation nor translation

**(5)** Motion about a fixed point






## (1) Translation

Consider rigid body in translation:

direction of any straight line inside the body is constant, all particles forming the body move in parallel lines.

For any two particles in the body, $r_B = r_A + r_{B/A}$

Differentiating with respect to time,

$$\dot{r}_B = \dot{r}_A + \dot{r}_{B/A} = \dot{r}_A$$

$$\dot{r}_{B/A} = 0$$

Magnitude constant since A, B belong to same rigid body

$$v_B = v_A$$

All particles have the same velocity.

Differentiating with respect to time again,

$$a_B = a_A$$

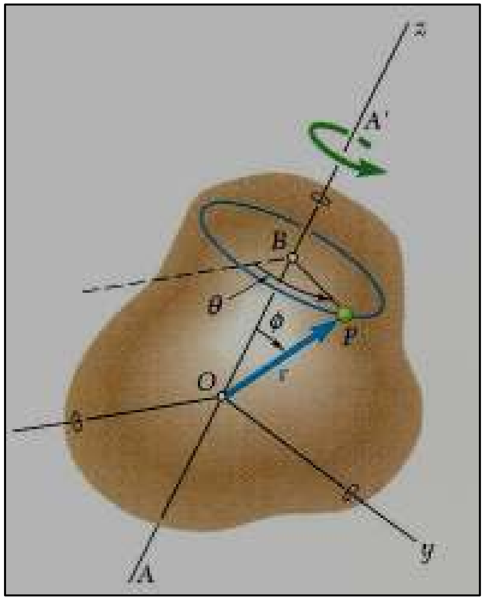
All particles have the same acceleration.

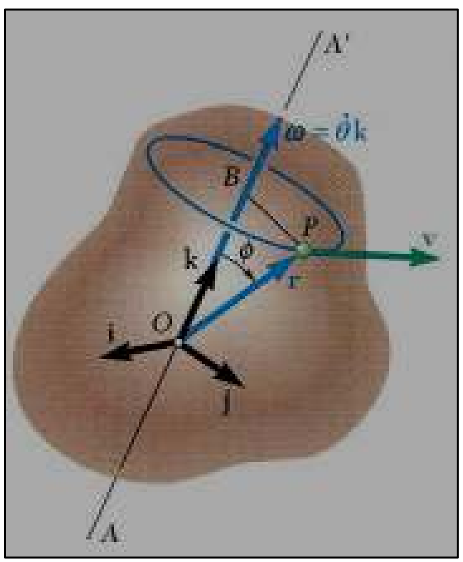
$$v_B = v_A \qquad a_B = a_A$$

When a rigid body is in translation, all the points of the body have the same velocity and same acceleration at any given instant

In curvilinear translation, the velocity and acceleration change in direction and in magnitude at every instant; In rectilinear translation, velocity, acceleration direction are same during entire motion

# (2) Rotation About a Fixed Axis: Velocity & Acceleration




Consider rotation of rigid body about a fixed axis *AA'*

The length Δs of the arc described by P when the body rotates through Δθ,

$$\Delta s = (BP)\Delta\theta = (r\sin\phi)\Delta\theta$$

$$v = \frac{ds}{dt} = \lim_{\Delta t \to 0} (r\sin\phi)\frac{\Delta\theta}{\Delta t} = r\dot{\theta}\sin\phi$$

$$\mathbf{V} = \frac{dr}{dt} = \omega \times r$$

Vector direction along the rotation axis AA', **angular velocity,** $\omega k = \dot{\theta} k$

- Differentiating to determine the acceleration,

$$a = \frac{dv}{dt} = \frac{d}{dt}(\omega \times r)$$

$$= \frac{d\omega}{dt} \times r + \omega \times \frac{dr}{dt}$$

$$= \frac{d\omega}{dt} \times r + \omega \times v$$

↓ $\frac{d\omega}{dt}$: Angular acceleration, α

$$a = \underline{\alpha \times r} + \underline{\omega \times (\omega \times r)}$$

↓ $\alpha \times r$: Tangential acceleration component

↓ $\omega \times (\omega \times r)$: Radial acceleration component

# Equations Defining the Rotation of a Rigid Body About a Fixed Axis

$$\omega = \frac{d\theta}{dt} \quad \text{or} \quad dt = \frac{d\theta}{\omega}$$

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} = \omega \frac{d\omega}{d\theta}$$

$v = dx/dt$

$a = d^2x/dt^2$

$a = v\,(dv/dx)$

## Uniform Rotation, $\alpha = 0$:

$$\theta = \theta_0 + \omega t$$

Equns. Can be used only when $\alpha = 0$ & constant

$x = x_0 + vt$

## Uniformly Accelerated Rotation, $\alpha$ = constant:

$$\omega = \omega_0 + \alpha t$$

$$\theta = \theta_0 + \omega_0 t + \tfrac{1}{2}\alpha t^2$$

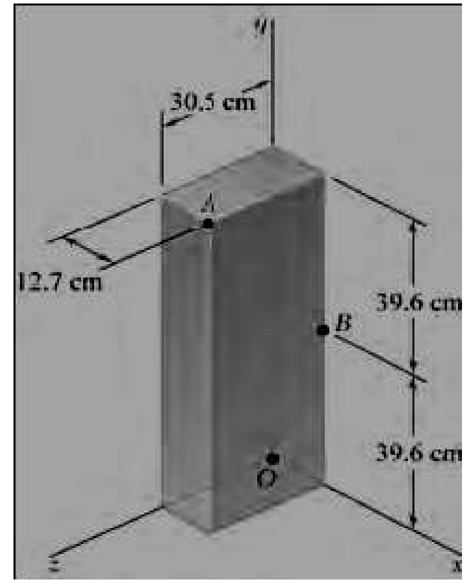
$$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$$

$v = v_0 + at$

$x = x_0 + v_0 t + ½ at^2$

$v^2 = v_0^2 + 2a\,(x - x_0)$

The rectangular block shown rotates about the diagonal OA with a constant angular velocity of 6.76 rad/s. Knowing that the rotation is counterclockwise as viewed from A, determine the velocity and acceleration of point B at the instant shown.




$$l_{OA} = \sqrt{(.127)^2 + (.792)^2 + (.305)^2} = .858 \text{ m}$$

Angular velocity.

$$\boldsymbol{\omega} = \frac{\omega}{l_{OA}} \mathbf{r}_{A/O} = \frac{6.76}{.858}(.127\,\mathbf{i} + .792\,\mathbf{j} + .305\,\mathbf{k})$$

$$\boldsymbol{\omega} = (1.0 \text{ rad/s})\mathbf{i} + (6.24 \text{ rad/s})\mathbf{j} + (2.4 \text{ rad/s})\mathbf{k}$$

Velocity of point B.

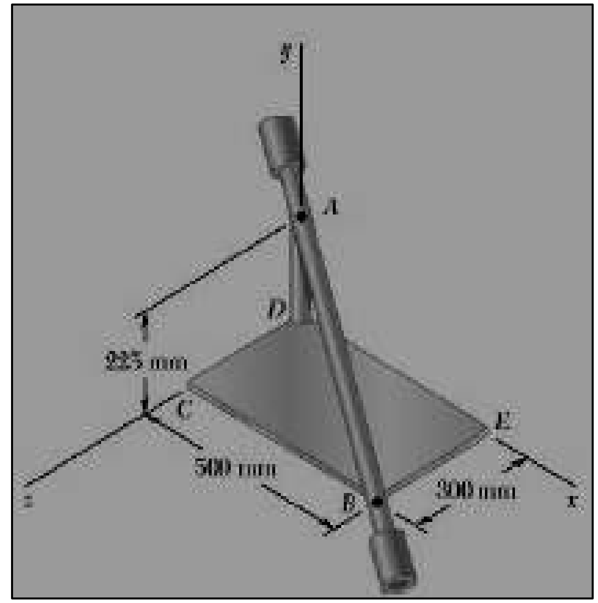
$$vB = \omega \times R_{B/O} = (i + 6.24j + 2.4\,k) \times (0.127i + 0.396j)$$

$$= -0.95i + 0.305j - 0.396\,k \qquad \text{m/s}$$

Acceleration of point B.

$$aB = \omega \times vB = -3.2\,i - 1.88j + 6.23k \qquad \text{m/s}^2$$

The assembly shown consists of two rods and a rectangular plate BCDE which are welded together. The assembly rotates about the axis AB with a constant angular velocity of 10 rad/s. Knowing that the rotation is counterclockwise as viewed from B, determine the velocity and acceleration of corner E.




Find $r_{B/A}$, $r_{E/B}$

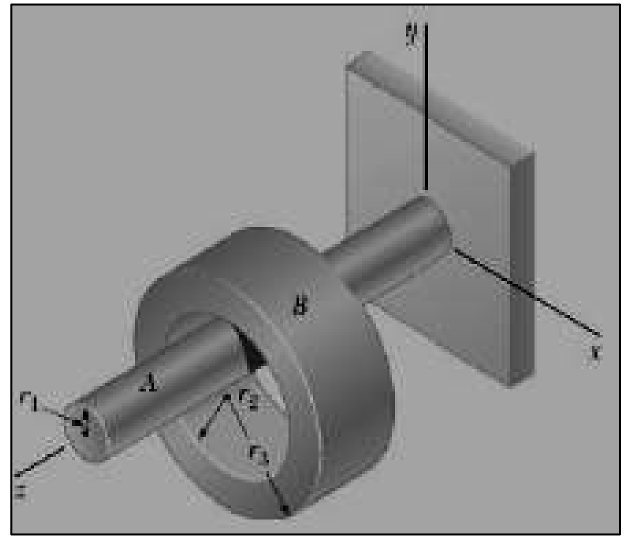
Find angular velocity vector; $\overline{\omega} = \omega\,(r_{B/A} / l_{AB})$

1) $v_E = \omega \times r_{E/B}$; 2) $a_E = \omega \times v_E$

$$\mathbf{v}_E = (1.080\ \text{m/s})\mathbf{i} + (2.40\ \text{m/s})\mathbf{j}$$

$$\mathbf{a}_E = -\left(11.52\ \text{m/s}^2\right)\mathbf{i} + \left(5.18\ \text{m/s}^2\right)\mathbf{j} + \left(23.1\ \text{m/s}^2\right)\mathbf{k}$$

Ring B has an inner radius r2 and hangs from the horizontal shaft A as shown. Shaft A rotates with a constant angular velocity of 25 rad/s and no slipping occurs. Knowing that r1 = 1.27 cm, r2 = 6.35 cm, and r3 = 8.9 cm, determine (a) the angular velocity of ring B, (b) the acceleration of the points of shaft A and ring B which are in contact, (c) the magnitude of the acceleration of a point on the outside surface of ring B.




(a) Let point $C$ be the point of contact between the shaft and the ring.

$$v_C = r_1\omega_A = (.0127)(25) = .3175 \text{ m/s}$$

$$\omega_B = \frac{v_C}{r_2} = \frac{.3175}{.0635} = 5.0 \text{ rad/s} \qquad \omega_B = 5.00 \text{ rad/s}$$

(b) On shaft $A$:

$$a_A = r_1\omega_A^2 = (.0127)(25)^2$$

$$= 8 \text{ m/s}^2, \qquad \mathbf{a}_A = 26.0 \text{ ft/s}^2$$

On ring $B$:

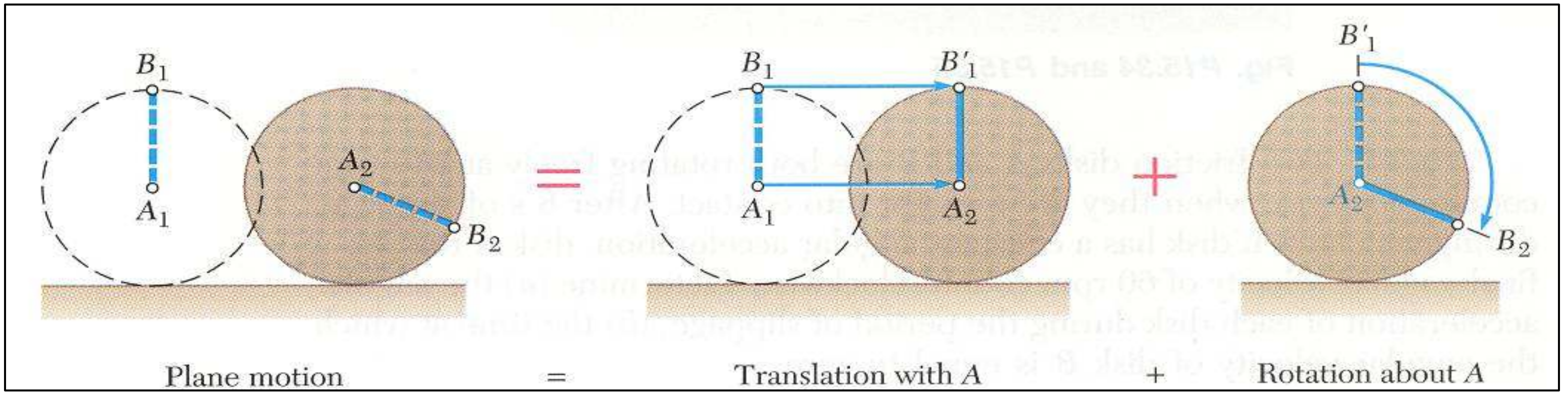
$$a_B = r_2\omega_B^2 = (.0635)(5.0)^2$$

$$= 1.58 \text{ m/s}^2, \qquad \mathbf{a}_B = 1.58 \text{ m/s}^2$$

$(c)$ At a point on the outside of the ring, $r = r_3 = .089\ \text{m}$

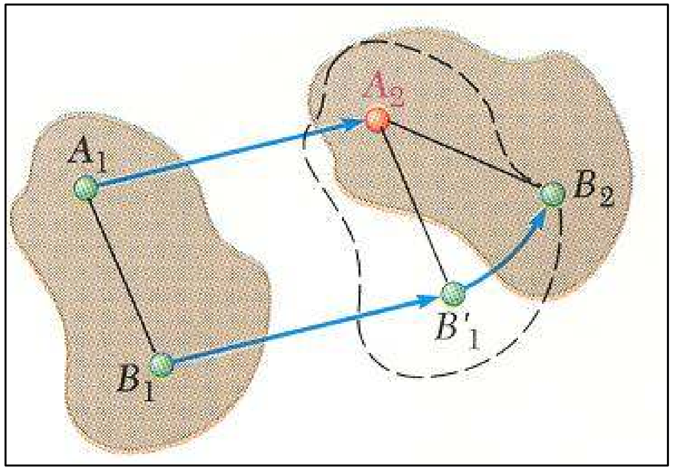
$$a = r\omega_B^2 = (.089)(5.0)^2 = 2.225\ \text{m/s}^2 \qquad a = 2.225\ \text{m/s}^2$$

# (3) General Plane Motion

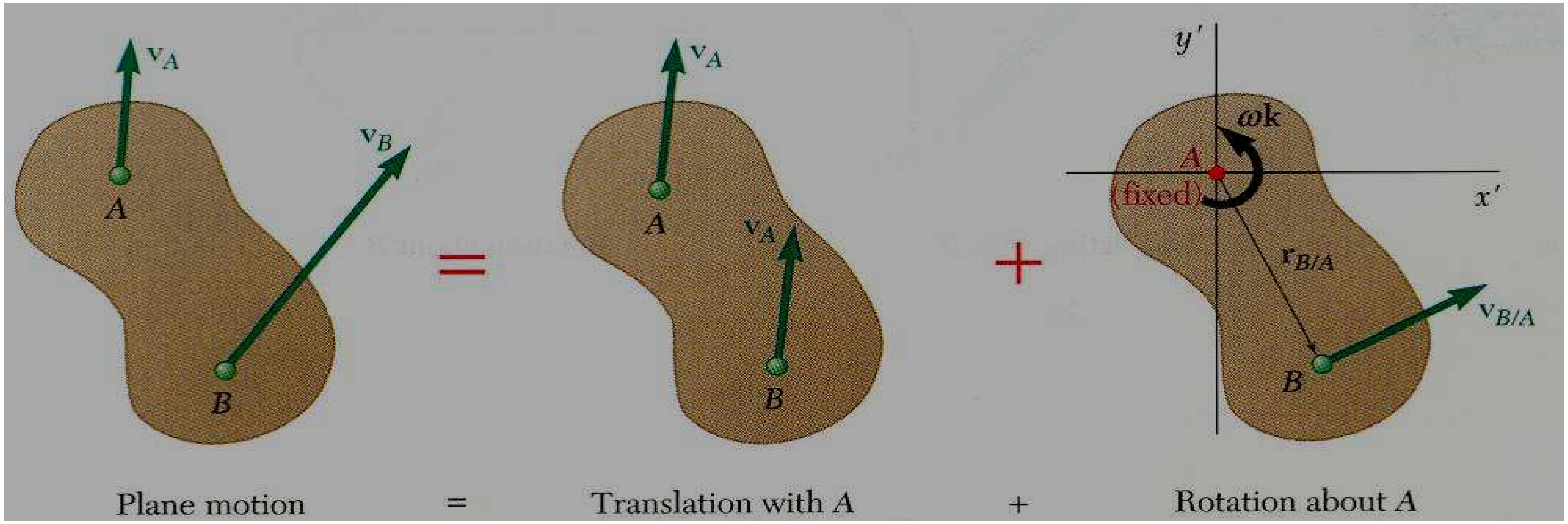



- General plane motion is neither a translation nor a rotation.
- General plane motion can be considered as the sum of a translation and rotation.

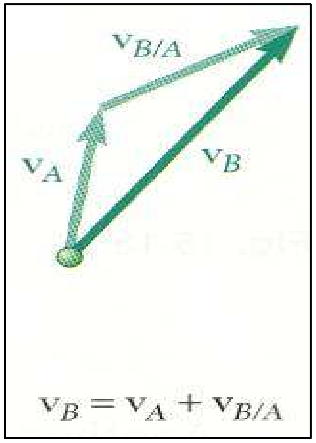
- Displacement of particles $A_1$ and $B_1$ to $A_2$ and $B_2$ can be divided into two parts:
  - translation to $A_2$ and $B_1'$
  - rotation of $B_1'$ about $A_2$ to $B_2$



# Absolute and Relative Velocity in Plane Motion




- Any plane motion can be replaced by a translation of an arbitrary reference point A and a simultaneous rotation about A.

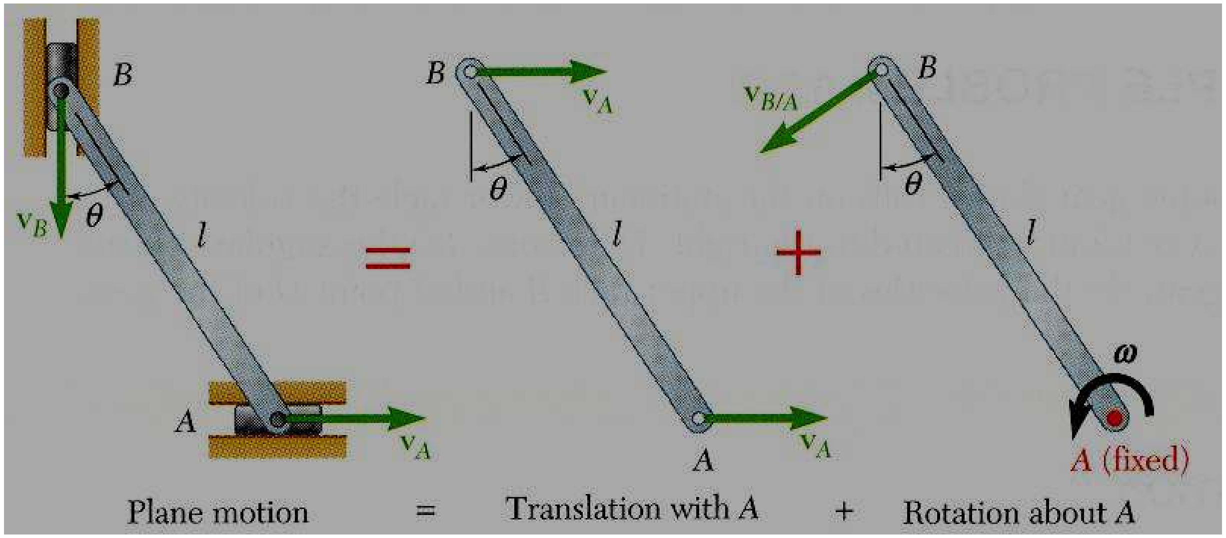


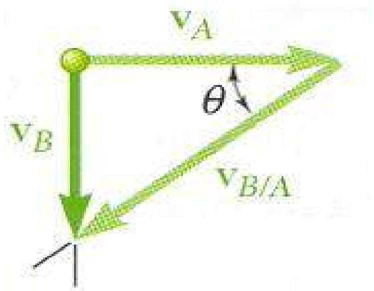

$$v_B = v_A + v_{B/A}$$

$$v_{B/A} = \overline{\omega} \times r_{B/A} \qquad v_{B/A} = r\omega$$

$$v_B = v_A + \overline{\omega} \times r_{B/A}$$

# Point A as reference




- Assuming that the velocity $v_A$ of end $A$ is known, wish to determine the velocity $v_B$ of end $B$ and the angular velocity ω in terms of $v_A$, $l$, and θ.

- The direction of $v_B$ and $v_{B/A}$ are known. Complete the velocity diagram.

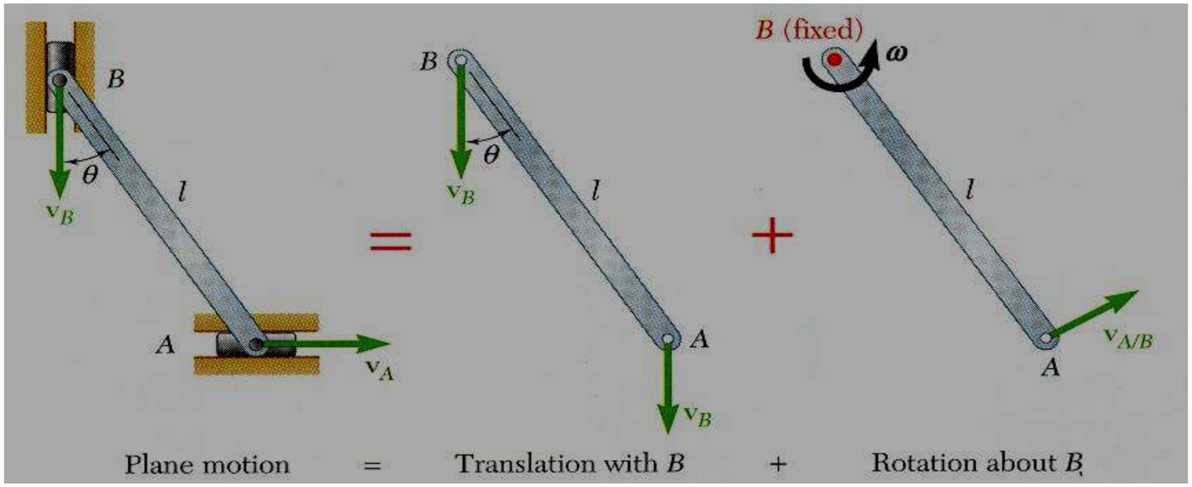
$$\frac{v_B}{v_A} = \tan\theta$$

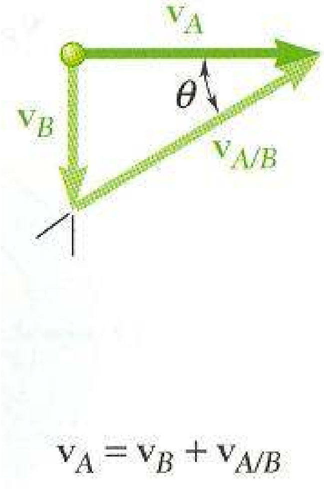
$$v_B = v_A \tan\theta$$

$$\frac{v_A}{v_{B/A}} = \frac{v_A}{l\omega} = \cos\theta$$

$$\omega = \frac{v_A}{l\cos\theta}$$

# Point B as reference

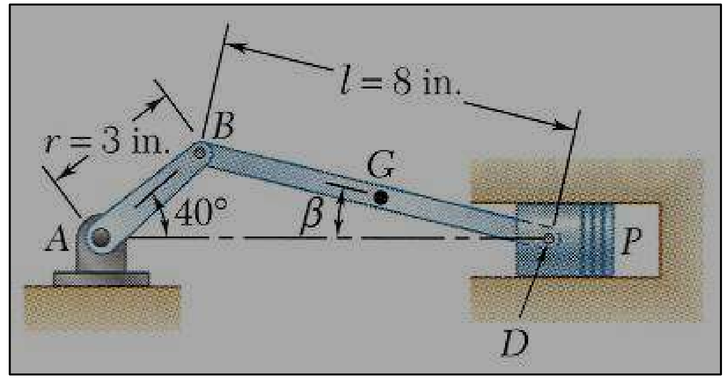


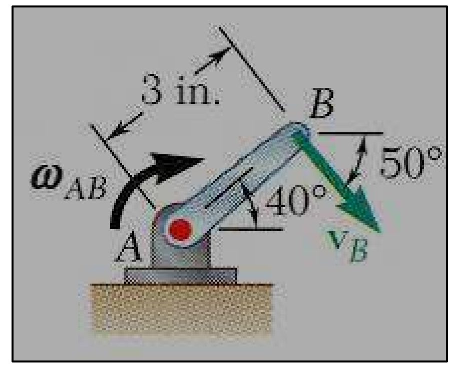

- Selecting point *B* as the reference point and solving for the velocity $v_A$ of end *A* and the angular velocity $\omega$ leads to an equivalent velocity triangle.
- $v_{A/B}$ has the same magnitude but opposite sense of $v_{B/A}$. The sense of the relative velocity is dependent on the choice of reference point.
- Angular velocity $\omega$ of the rod in its rotation about *B* is the same as its rotation about *A*. Angular velocity is not dependent on the choice of reference point.

The crank AB has a constant clockwise angular velocity of 2000 rpm.

For the crank position indicated, determine (a) the angular velocity of the connecting rod BD, and (b) the velocity of the piston P.




- The velocity $\vec{v}_B$ is obtained from the crank rotation data.




**Crank AB**

$$\omega_{AB} = \left(2000\frac{\text{rev}}{\text{min}}\right)\left(\frac{\text{min}}{60\,\text{s}}\right)\left(\frac{2\pi\,\text{rad}}{\text{rev}}\right) = 209.4\,\text{rad/s}$$

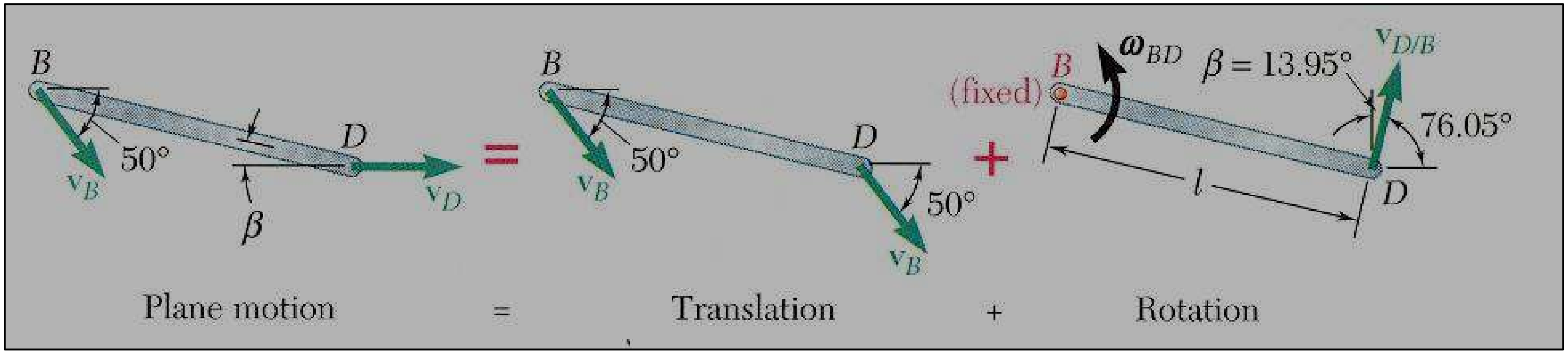
$$v_B = (AB)\omega_{AB} = (3\,\text{in.})(209.4\,\text{rad/s}) \quad = 628.3\ \text{in/s}$$

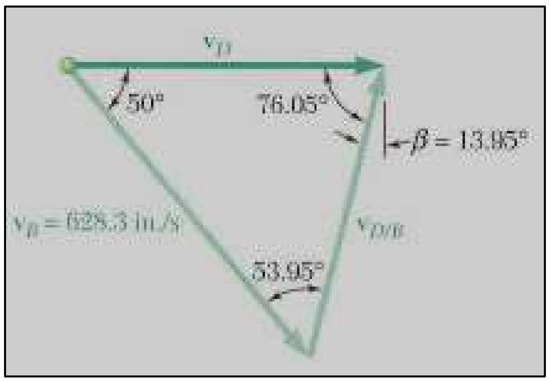
The velocity direction is as shown.

**Connecting rod BD**

- The direction of the absolute velocity $v_D$ is horizontal. The direction of the relative velocity $v_{D/B}$ is perpendicular to *BD*. Compute the angle between the horizontal and the connecting rod from the law of sines.

$$\frac{\sin 40°}{8\,\text{in.}} = \frac{\sin\beta}{3\,\text{in.}} \qquad \beta = 13.95°$$




$$\frac{v_D}{\sin 53.95^\circ} = \frac{v_{D/B}}{\sin 50^\circ} = \frac{628.3\,\text{in./s}}{\sin 76.05^\circ}$$

$$v_D = 523.4\,\text{in./s} = 43.6\,\text{ft/s}$$

$$v_P = v_D = 43.6\,\text{ft/s}$$

$$v_{D/B} = 495.9\,\text{in./s}$$

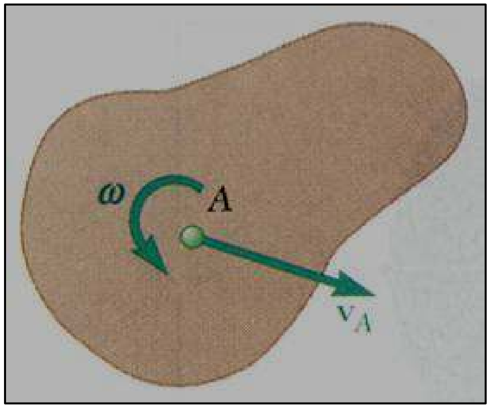
$$v_{D/B} = l\omega_{BD}$$

$$\omega_{BD} = \frac{v_{D/B}}{l} = \frac{495.9\,\text{in./s}}{8\,\text{in.}}$$

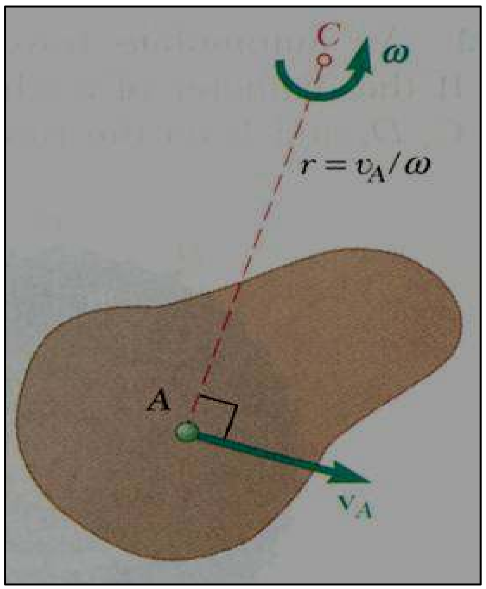
$$= 62.0\,\text{rad/s}$$

$$\omega_{BD} = (62.0\,\text{rad/s})k$$

# Instantaneous Center of Rotation in Plane Motion

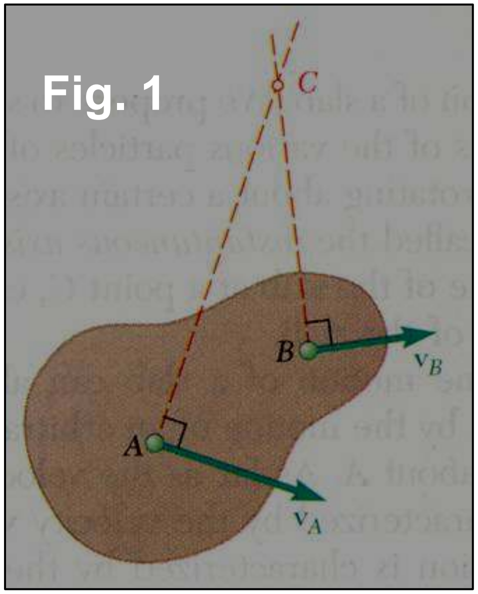


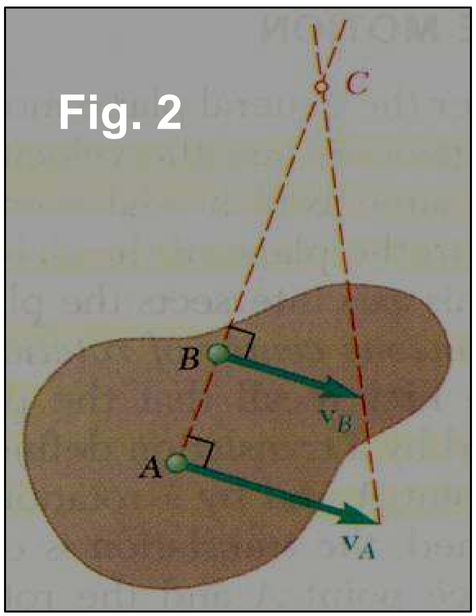

- Plane motion of all particles in a slab can always be replaced by the translation of an arbitrary point *A* and a rotation about *A* with an angular velocity that is independent of the choice of *A*.




- The same translational and rotational velocities at *A* are obtained by allowing the slab to rotate with the same angular velocity about the point *C* on a perpendicular to the velocity at *A*.
- The velocity of all other particles in the slab are the same as originally defined since the angular velocity and translational velocity at *A* are equivalent.
- As far as the velocities are concerned, the slab seems to rotate about the *instantaneous center of rotation C*.

# How to obtain instantaneous center of rotation



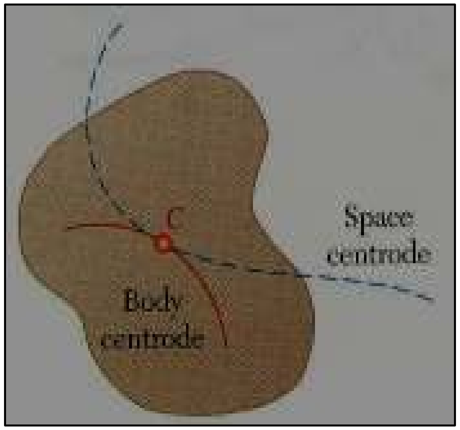




- If the velocity at two points *A* and *B* are known, the instantaneous center of rotation lies at the intersection of the perpendiculars to the velocity vectors through *A* and *B* . **(fig. 1)**
- If the velocity vectors at *A* and *B* are perpendicular to the line *AB*, the instantaneous center of rotation lies at the intersection of the line *AB* with the line joining the extremities of the velocity vectors at *A* and *B*. **(fig. 2)**

- If the velocity vectors are parallel, the instantaneous center of rotation is at infinity and the angular velocity is zero. **(fig. 1)**

- If the velocity magnitudes are equal, the instantaneous center of rotation is at infinity and the angular velocity is zero. **(fig. 2)**

- Instantaneous center of a slab in plane motion can be located either on slab or on outside the slab. If on the slab, the particle 'C' coinciding at the center of rotation has zero velocity at that instant. This coincidence will not happen at another time. So velocity at time 't' will not be same at t+dt.
- The particle coinciding with the center of rotation changes with time and the acceleration of the particle at the instantaneous center of rotation 'C' is not zero.
- The acceleration of the particles in the slab cannot be determined as if the slab were simply rotating about *C*.

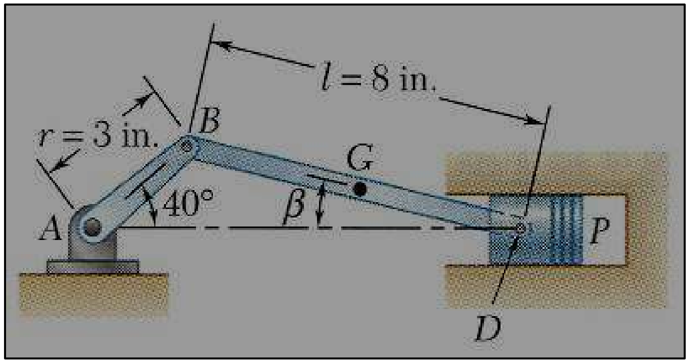



- The trace of the locus of the center of rotation on the body is the <u>body centrode</u> and in space is the <u>space centrode.</u>

## Same problem

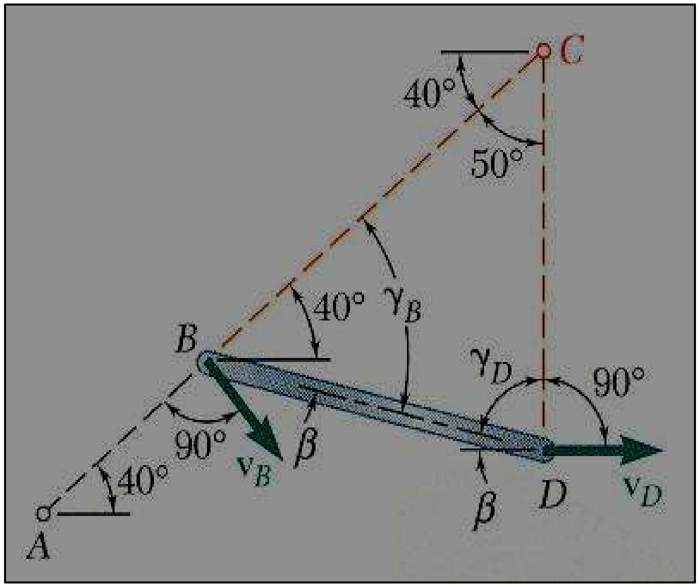
The crank *AB* has a constant clockwise angular velocity of 2000 rpm.

For the crank position indicated, determine *(a)* the angular velocity of the connecting rod *BD*, and *(b)* the velocity of the piston *P*.




vB = 628.3 in/s; β = 13.95°

- The instantaneous center of rotation is at the intersection of the perpendiculars to the velocities through *B* and *D*.




$$\gamma_B = 40° + \beta = 53.95°$$
$$\gamma_D = 90° - \beta = 76.05°$$

$$\frac{BC}{\sin 76.05°} = \frac{CD}{\sin 53.95°} = \frac{8 \text{ in.}}{\sin 50°}$$

$$BC = 10.14 \text{ in.} \quad CD = 8.44 \text{ in.}$$

- Determine the angular velocity about the center of rotation based on the velocity at *B*.

$$v_B = (BC)\omega_{BD}$$

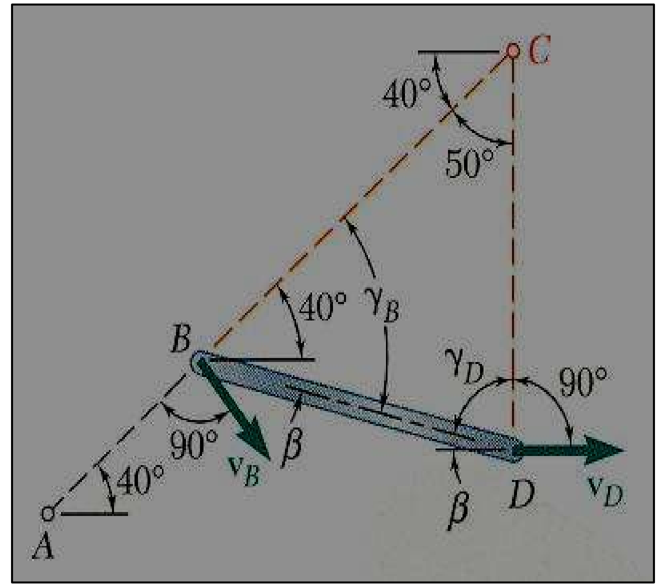
$$\omega_{BD} = \frac{v_B}{BC} = \frac{628.3\,\text{in./s}}{10.14\,\text{in.}}$$

$$\omega_{BD} = 62.0\,\text{rad/s}$$

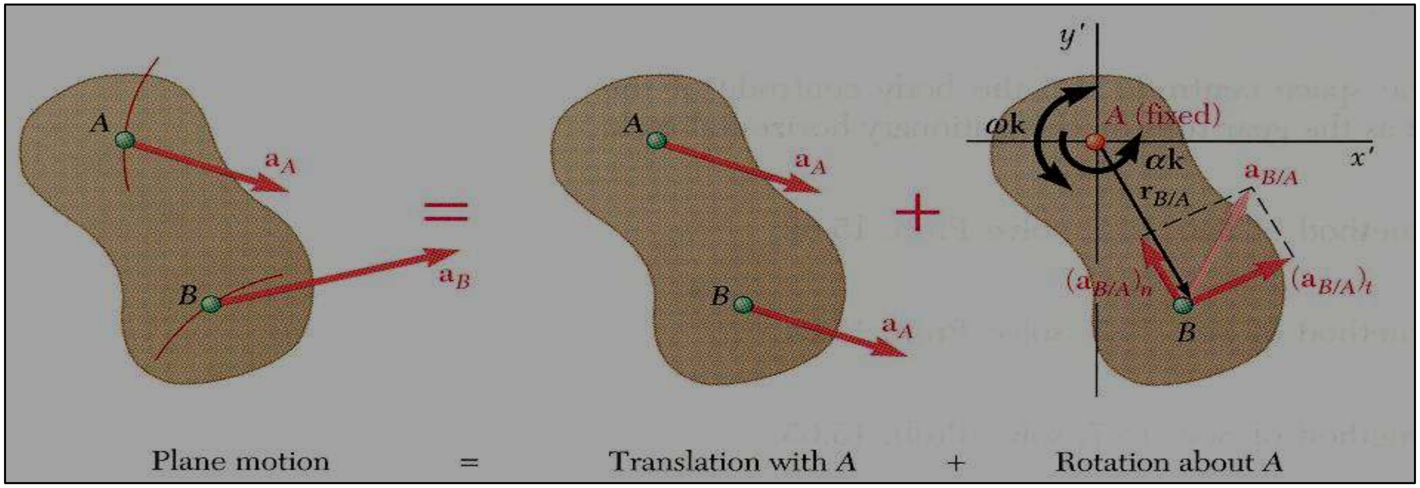
- Calculate the velocity at *D* based on its rotation about the instantaneous center of rotation.

$$v_D = (CD)\omega_{BD} = (8.44\,\text{in.})(62.0\,\text{rad/s})$$

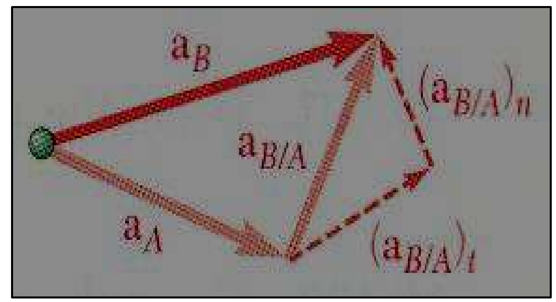
$$v_P = v_D = 523\,\text{in./s} = 43.6\,\text{ft/s}$$



# Absolute and Relative Acceleration in Plane Motion




- Absolute acceleration of a particle of the slab,




$$a_B = a_A + a_{B/A}$$

$$\begin{aligned} a_B &= a_A + a_{B/A} \\ &= a_A + \left(a_{B/A}\right)_n + \left(a_{B/A}\right)_t \end{aligned}$$

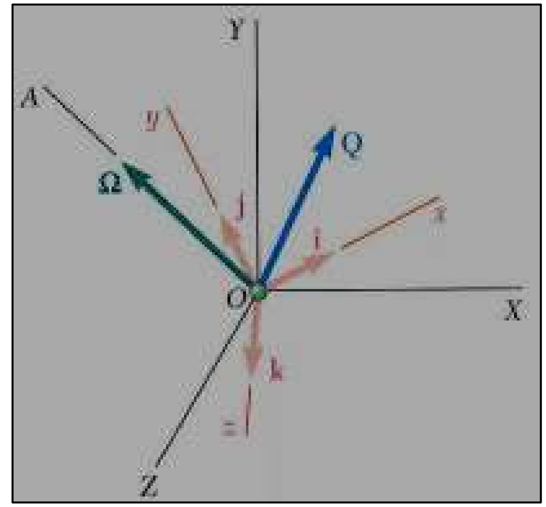
- Relative acceleration $\vec{a}_{B/A}$ associated with rotation about $A$ includes tangential and normal components,

$$\left(a_{B/A}\right)_t = \alpha \times r_{B/A} \qquad \left(a_{B/A}\right)_t = r\alpha$$

$$\left(a_{B/A}\right)_n = -\omega^2 r_{B/A} \qquad \left(a_{B/A}\right)_n = r\omega^2$$

# Rate of Change of vector with respect to a Rotating Frame

- Frame OXYZ is fixed.
- Frame *Oxyz* rotates about fixed axis *OA* with angular velocity $\Omega$
- Vector function $Q(t)$ varies in direction and magnitude.




Rate of change of Q depends on frame of reference

- With respect to the rotating Oxyz frame,

$$Q = Q_x i + Q_y j + Q_z k$$

$$(\dot{Q})_{Oxyz} = \dot{Q}_x i + \dot{Q}_y j + \dot{Q}_z k$$

- If $Q$ were fixed within *Oxyz* then $(\dot{Q})_{OXYZ}$ is equivalent to velocity of a point in a rigid body attached to *Oxyz* and

$$Q_x di/dt + Q_y\, dj/dt + Q_z dk/dt = \Omega \times Q$$

- With respect to the fixed *OXYZ* frame,

$$(\dot{Q})_{OXYZ} = \dot{Q}_x i + \dot{Q}_y j + \dot{Q}_z k + Q_x d\mathbf{i}/dt + Q_y\, d\mathbf{j}/dt + Q_z d\mathbf{k}/dt$$

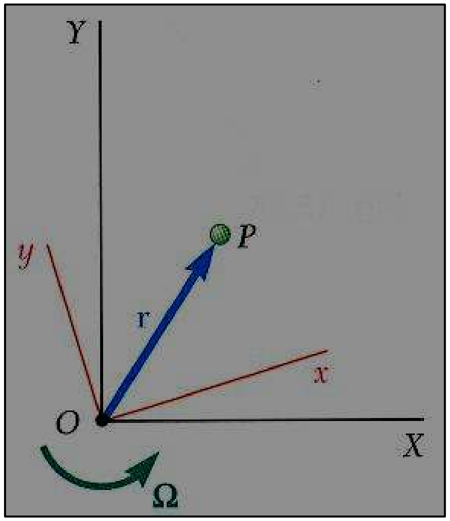
$$(\dot{Q})_{Oxyz} = \dot{Q}_x i + \dot{Q}_y j + \dot{Q}_z k$$

Represent velocity of particle, ΩxQ

- With respect to the fixed *OXYZ* frame,

$$\left(\dot{Q}\right)_{OXYZ} = \left(\dot{Q}\right)_{Oxyz} + \Omega \times Q$$

This relation is useful to find rate of change of Q w.r.t. fixed frame of reference OXYZ when Q is defined by its components along the rotating frame Oxyz




# Coriolis Acceleration

- Frame *OXY* is fixed and frame *Oxy* rotates with angular velocity $\vec{\Omega}$.
- Position vector $r_P$ for the particle *P* is the same in both frames but the rate of change depends on the choice of frame.
- The absolute velocity of the particle *P* is, $v_P = (\dot{r})_{OXY} = \Omega \times r + (\dot{r})_{Oxy}$
- The absolute acceleration of the particle *P* is,

$$a_p = \dot{v}_p = \dot{\Omega} x r + \Omega x \dot{r} + \underline{d/dt\, [(\dot{r})_{Oxy}]}$$

but, $(\dot{\vec{r}})_{OXY} = \vec{\Omega} \times \vec{r} + (\dot{\vec{r}})_{Oxy}$

$$v_P = (\dot{r})_{OXY} = \Omega \times r + (\dot{r})_{Oxy}$$

$$\frac{d}{dt}\left[(\dot{\vec{r}})_{Oxy}\right] = (\ddot{\vec{r}})_{Oxy} + \vec{\Omega} \times (\dot{\vec{r}})_{Oxy}$$

$$a_p = \dot{\Omega} \text{ x } r + \Omega \text{ x } (\Omega \text{ x } r) + 2\Omega \text{ x } (\dot{r})_{Oxy} + (\ddot{r})_{Oxy}$$

$$a_p = \dot{\Omega} \times r + \Omega \times (\Omega \times r) + 2\Omega \times (\dot{r})_{Oxy} + (\ddot{r})_{Oxy}$$

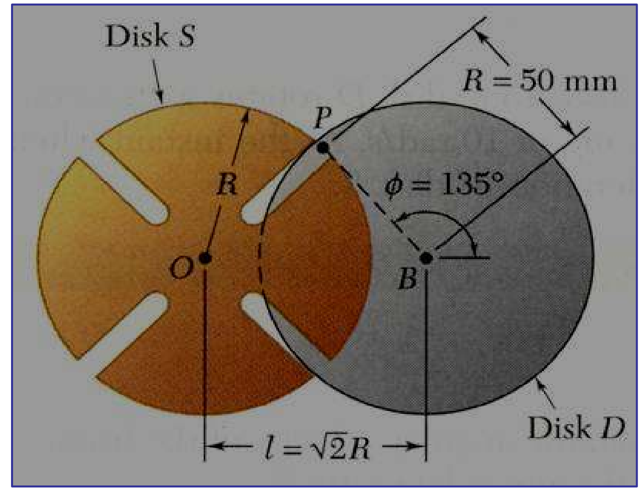
$a = \alpha \times r + \omega \times (\omega \times r)$

Coriolis acceleration, $a_c$

Two vectors are normal to each other, $2\Omega v_{Oxy}$

Disk D of the Geneva mechanism rotates with constant counterclockwise angular velocity $\omega_D$ = 10 rad/s.

At the instant when $\phi$= 150$^o$, determine (*a*) the angular velocity of disk *S*, and (*b*) the velocity of pin *P* relative to disk *S*.

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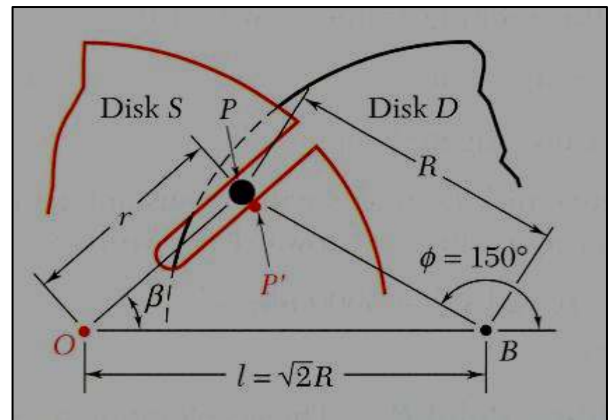
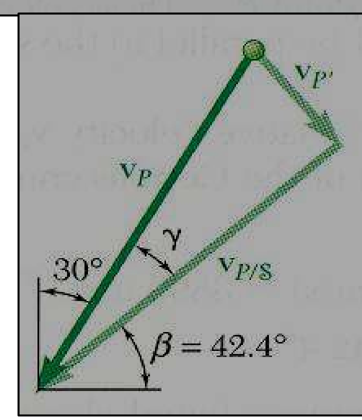
From the law of cosines,

$$r^2 = R^2 + l^2 - 2Rl\cos 30° = 0.551R^2 \qquad r = 37.1\,\text{mm}$$




From the law of sine,

$$\frac{\sin\beta}{\text{R}} = \frac{\sin 30°}{r} \qquad \sin\beta = \frac{\sin 30°}{0.742} \qquad \beta = 42.4°$$



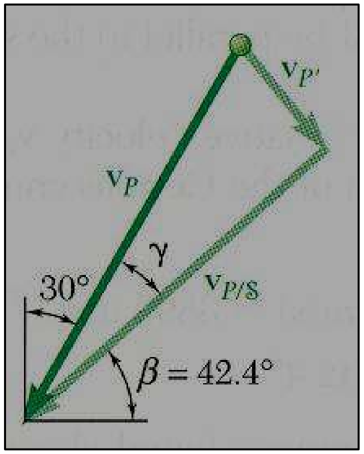

- Magnitude and direction of absolute velocity of pin *P* are calculated from radius and angular velocity of disk *D*.

$$v_P = R\omega_D = (50\text{ mm})(10\text{ rad/s}) = 500\text{mm/s}$$

- The absolute velocity of the point *P* may be written as

$$v_P = (\dot{r})_{OXY} = \Omega \times r + (\dot{r})_{Oxy}$$

$$v_P = v_{P'} + v_{P/s}$$




The interior angle of the vector triangle is

$$\gamma = 90° - 42.4° - 30° = 17.6°$$

$$v_{P'} = v_P \sin\gamma = (500\,\text{mm/s})\sin 17.6° = 151.2\,\text{mm/s}$$

$$= r\omega_s \qquad \omega_s = \frac{151.2\,\text{mm/s}}{37.1\,\text{mm}}$$

$$\omega_s = 4.08\,\text{rad/s} \curvearrowright$$

$$v_{P/s} = v_P \cos\gamma = (500\,\text{m/s})\cos 17.6°$$

$$v_{P/s} = (477\,\text{m/s}) \quad 42.4° \swarrow$$

Coriolis acceleration, ac = 2 $\omega_s$ $v_{p/s}$ = 2 (4.08) (477) = 3890 mm/s$^2$